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SPACES OF BMO TYPE

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Abstract. It is presented a Banach space of functions of bounded mean oscillation BMO type.

1. Introduction

The space of functions of bounded mean oscillation, or BMO, naturally arises as a class of functions whose deviation from their means over cubes is bounded. In fact, the classical BMO-norm $\|f\|_{BMO}$ for the equivalent class of a locally integrable function f on $\mathbb{R}^d$ ($f \in L^1_{loc}(\mathbb{R}^d)$) is defined as

$$\|f\|_{BMO} := \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx \tag{1.1}$$

for every cube $Q \subset \mathbb{R}^d$ whose sides are parallel to the axes and

$$f_Q := \frac{1}{|Q|} \int_Q f(x) dx.$$

The space BMO first appeared in the work of John and Nirenberg [1] in the context of nonlinear partial differential equations that emerge in the study of minimal surfaces.

Even though the Lebesgue space L^∞ functions have the same property, there exist unbounded functions with bounded mean oscillation. Such functions are slowly growing, and typically have at most logarithmic blow-up. The space BMO shares similar properties with the space L^∞, and it often serves as a substitute for it. For instance, classical singular integrals do not map L^∞ to L^∞ but L^∞ to BMO. In many instances the interpolation between L^p and BMO works just as well

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between L^p and L^∞. Indeed, the role of the space BMO is deeper and more far-reaching than that [6]. This space crucially arises in many situations in analysis, such as in the characterization of the L^2-boundedness of non-convolution singular integral operators with standard kernels.

Recently, we have built up a new function space in order to generalize the classical Lebesgue spaces [3, 4, 5]. The motivation of this research stems from taking a close look at the L^p-norm: $\|f\|_{L^p} = \left(\int_X |f(x)|^p \, d\mu\right)^{1/p}$ of the Lebesgue spaces $L^p(X)$, $1 \le p < \infty$. It can be rewritten as

$$\|f\|_{L^p} := \alpha^{-1}\left(\int_X \alpha(|f(x)|)\, d\mu\right), \quad f \in L^p(X) \tag{1.2}$$

with the base function α as

$$\alpha(x) := x^p.$$

By virtue of the John-Nirenberg inequality, it is well-known that the classical BMO-norm (1.1) is equivalent to the L^p characterization of BMO-norm $\|\cdot\|_{BMO_p}$ defined by

$$\|f\|_{BMO_p} := \left(\sup_Q \frac{1}{|Q|}\int_Q |f(x)-f_Q|^p dx\right)^{\frac{1}{p}}, \quad (f \in L^1_{loc}(\mathbb{R}^d)) \tag{1.3}$$

for $1 < p < \infty$.

In the same line of our research we introduce a functional

$$\|f\|_{BMO_\alpha} := \alpha^{-1}\left(\sup_Q \frac{1}{|Q|}\int_Q \alpha(|f(x)-f_Q|)dx\right), \ (f \in L^1_{loc}(\mathbb{R}^d)) \tag{1.4}$$

for an appropriate base function α. The main point of this report is to present sufficient conditions of base functions α such that $\|\cdot\|_{BMO_\alpha}$ forms a (quisi-)norm, so it constitutes a natural Banach space BMO_α.

The base functions α which we have developed include base functions of the form $\alpha(x) = x^p$, and we designate the base functions α to achieve the Minkowski type triangle inequality. This research was inspired by [2].

2. The main theorem and arguments

We have been developed appropriate base functions that permit the Hölder type inequality. In this section, we briefly introduce the concepts of admissable base functions - the details can be found in [3, 4]. The notions presented here are modified versions without essential differences. In the following, $\bar{\mathbb{R}}_+$ represents $\{x \in \mathbb{R} : x \geq 0\}$.

Let $\alpha, \beta : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ be strictly increasing absolutely continuous functions. The pair (α, β) is called a *pre-Hölder* pair if it obeys

$$\alpha^{-1}(x)\beta^{-1}(x) = x \tag{2.1}$$

for all $x \in \bar{\mathbb{R}}_+$. In the relation (2.1), the notations α^{-1}, β^{-1} are the inverse functions of α, β, respectively. Some examples of pre-Hölder pairs are:

$$(\alpha(x), \beta(x)) = (x^p, x^q)$$

for $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and

$$(\alpha, \beta) := (\lambda \circ A, \lambda \circ \tilde{A}) \tag{2.2}$$

where we set $\lambda(x) = A^{-1}(x)\tilde{A}^{-1}(x)$ for any Orlicz N-function A together with its complementary N-function $\tilde{A}$.

In the following, Q stands for a cube whose sides are parallel to the axes and $|A|$ is the Lebesgue measure of the set A in $\mathbb{R}^d$, $d \geq 1$. We state the main theorem.

THEOREM 2.1. *Let $\hbar > 0$ be given. Suppose that (α, β) is a pre-Hölder pair such that for any positive constants $a, b > 0$, there exist constants θ_1, θ_2 and θ_f (depending on a and b) satisfying the following two conditions;*

$$\theta_1 + \theta_2 + \theta_f \leq \hbar$$

and

$$\alpha^{-1}(x)\beta^{-1}(y) \leq \theta_1 \frac{ab}{\alpha(a)}\, x + \theta_2 \frac{ab}{\beta(b)}\, y + ab\,\theta_f \tag{2.3}$$

for all $(x, y) \in \bar{\mathbb{R}}_+ \times \bar{\mathbb{R}}_+$. Then the functional

$$\|f\|_{BMO_\alpha} := \alpha^{-1}\left(\sup_Q \frac{1}{|Q|}\int_Q \alpha(|f(x) - f_Q|)dx\right) \tag{2.4}$$

satisfies a Minkowski type inequality : for any locally integrable functions f and g, we have

$$\|f + g\|_{BMO_\alpha} \leq \hbar\{\|f\|_{BMO_\alpha} + \|g\|_{BMO_\alpha}\} \tag{2.5}$$

if the right hand side is finite. Also, for any constant $k \geq 0$ and for a locally integrable function f, we obtain

$$\frac{k}{\hbar}\|f\|_{BMO_\alpha} \leq \|kf\|_{BMO_\alpha} \leq k\hbar\|f\|_{BMO_\alpha}.$$

In particular, when $\hbar = 1$, we have the homogeneity:

$$\|kf\|_{BMO_\alpha} = k\|f\|_{BMO_\alpha}.$$

For example, any (convex) function satisfying

$$\alpha(x) := \begin{cases} x^p & \text{for } 0 \leq x \leq 1 \\ x^q & \text{for sufficiently large } x \end{cases} \tag{2.6}$$

$(1 < p, q < \infty)$ obeys the conditions in Theorem 2.1, and so are many variants of (2.6).

For a locally integrable function f on $\mathbb{R}^d$, we let

$$f_Q = \frac{1}{|Q|}\int_Q f(x)dx := \fint_Q f(x)dx.$$

Let α be a pre-Hölder function. We denote a class of functions by

$$BMO_\alpha(\mathbb{R}^d) = \left\{f \in L^1_{loc}(\mathbb{R}^d) : \|f\|_{BMO_\alpha} < \infty\right\},$$

where we set

$$\|f\|_{BMO_\alpha} = \alpha^{-1}\left(\sup_Q \fint_Q \alpha(|f(x) - f_Q|)dx\right).$$

When α is the identity function, we write $BMO_\alpha(\mathbb{R}^d) := BMO(\mathbb{R}^d)$. In the sequel, the elements of $BMO_\alpha(\mathbb{R}^d)$ whose difference is a constant are identified. Note that even though we define $\|\cdot\|_{BMO_\alpha}$ on abstract measure spaces, we restrict our attention to the Euclidean space $\mathbb{R}^d$ equipped with Lebesgue measure.

We now present the proof.

Proof of Theorem 2.1. We first present a Hölder type inequality: for any $f, g \in BMO_\alpha(\mathbb{R}^d)$, we have

(2.7)
$$\left|\fint_Q f(x)g(x)\,dx\right| \leq \hbar\alpha^{-1}\left(\sup_Q \fint_Q \alpha(|f(x)|)dx\right)\beta^{-1}\left(\sup_Q \fint_Q \beta(|g(x)|)dx\right),$$

where Q is a cube whose sides are parallel to the axes.

For the proof of (2.7), we may assume that the right hand side of (2.7) is finite. We put

$$a := \alpha^{-1}\left(\sup_Q \fint_Q \alpha(|f(x)|)dx\right), \quad b := \beta^{-1}\left(\sup_Q \fint_Q \beta(|g(x)|)dx\right).$$

Then there exist constants θ_1, θ_2 and θ_f such that $\theta_1+\theta_2+\theta_f \le \hbar$ and

$$\begin{aligned}
|f(x)g(x)| &= \alpha^{-1}(\alpha(|f(x)|))\beta^{-1}(\beta(|g(x)|)) \\
&\le \theta_1 \frac{ab}{\alpha(a)}\,\alpha(|f(x)|) + \theta_2 \frac{ab}{\beta(b)}\,\beta(|g(x)|) + ab\theta_f.
\end{aligned} \tag{2.8}$$

Integrating over Q and dividing both sides by $|Q|$ yield

$$\begin{aligned}
\fint_Q |f(x)g(x)|\,dx \le \theta_1 \frac{ab}{\alpha(a)} \fint_K \alpha(|f(x)|)\,dx + \theta_2 \frac{ab}{\beta(b)} \fint_Q \beta(|g(x)|)\,dx \\
+ \theta_f ab \fint_Q dx \\
\le \hbar\alpha^{-1}\left(\sup_Q \fint_Q \alpha(|f(x)|)dx\right)\beta^{-1}\left(\sup_Q \fint_Q \beta(|g(x)|)dx\right).
\end{aligned}$$

This implies the Hölder type inequality (2.7).

We now verify the Minkowski type inequality (2.5). In fact, without loss of generality, we may assume that $f(x)+g(x) \neq 0$ almost every $x \in \mathbb{R}^d$ by restricting the domain $\mathbb{R}^d$ if necessary. Applying Hölder type inequality (2.7), we obtain

$$\begin{aligned}
&⨍_Q \alpha(|f(x)+g(x)-f_Q-g_Q|)\,dx \\
&\le \hbar\ \alpha^{-1}\left(\sup_Q ⨍_Q \alpha(|f(x)-f_Q|)\,dx\right) \\
&\qquad \times \beta^{-1}\left(\sup_Q ⨍_Q \beta\left(\frac{\alpha(|f(x)-f_Q+g(x)-g_Q|)}{|f(x)-f_Q+g(x)-g_Q|}\right)dx\right) \\
&\quad + \hbar\ \beta^{-1}\left(\sup_Q ⨍_Q \beta(|g(x)-g_Q|)\,dx\right) \\
&\qquad \times \beta^{-1}\left(\sup_Q ⨍_Q \beta\left(\frac{\alpha(|f(x)-f_Q+g(x)-g_Q|)}{|f(x)-f_Q+g(x)-g_Q|}\right)dx\right) \\
&= \hbar(\|f\|_{BMO_\alpha}+\|g\|_{BMO_\alpha}) \\
&\qquad \times \beta^{-1}\left(\sup_Q ⨍_Q \beta\left(\frac{\alpha(|f(x)-f_Q+g(x)-g_Q|)}{|f(x)-f_Q+g(x)-g_Q|}\right)dx\right) \\
&= \hbar(\|f\|_{BMO_\alpha}+\|g\|_{BMO_\alpha})\beta^{-1}\left(\sup_Q ⨍_Q \alpha(|f(x)-f_Q+g(x)-g_Q|)\,dx\right).
\end{aligned}$$

The last equality follows from the fact that

$$\alpha(x) = \beta\left(\frac{\alpha(x)}{x}\right). \tag{2.9}$$

In fact, we solve for $\beta^{-1}(x)$ in the conjugate identity $\alpha^{-1}(x)\beta^{-1}(x) = x$ to get $\beta^{-1}(x) = \dfrac{x}{\alpha^{-1}(x)}$, which in turn yields

$$x = \beta\left(\frac{x}{\alpha^{-1}(x)}\right).$$

This illustrates the identity (2.9). Therefore we obtain

$$\frac{\alpha(\|f+g\|_{BMO_\alpha})}{\beta^{-1}(\alpha(\|f+g\|_{BMO_\alpha}))} \le \ \|f\|_{BMO_\alpha} + \|g\|_{BMO_\alpha}. \tag{2.10}$$

From a variance of the conjugate identity: $\alpha^{-1}(x) = \dfrac{x}{\beta^{-1}(x)}$, we have

$$x = \frac{\alpha(x)}{\beta^{-1}(\alpha(x))}. \tag{2.11}$$

Hence from (2.10), we conclude the Minkowski type inequality (2.5):

$$\|f+g\|_{BMO_\alpha} \le \|f\|_{BMO_\alpha} + \|g\|_{BMO_\alpha}.$$

We now verify that for any constant $k \ge 0$ and for $f \in BMO_\alpha(\mathbb{R}^d)$, we have

$$\frac{k}{\hbar}\|f\|_{BMO_\alpha} \le \|kf\|_{BMO_\alpha} \le k\hbar\|f\|_{BMO_\alpha}. \tag{2.12}$$

For each $f \in BMO_\alpha(\mathbb{R}^d)$, the associated operator (inhomogeneous) norm of f is defined by

(2.13)
$$\|f\|_* := \sup\left\{ \frac{\left|\sup_Q ⨍_Q (f(x) - f_Q)g(x)dx\right|}{\beta^{-1}\left(\sup_Q ⨍_Q \beta(|g(x)|)dx\right)} : g(x) \ne 0 \text{ almost everywhere} \right\}.$$

Then we note that for any constant $k \ge 0$,

$$\|kf\|_* = k\|f\|_* \tag{2.14}$$

and by virtue of the Hölder type inequality (2.7), we have

$$\frac{\left|\sup_Q ⨍_Q (f(x) - f_Q)g(x)dx\right|}{\beta^{-1}\left(\sup_Q ⨍_Q \beta(|g(x)|)dx\right)} \le \hbar\|f\|_{BMO_\alpha}$$

for each measurable function g with $g(x) \ne 0$ almost everywhere. On the other hand, taking

$$g(x) := \frac{\alpha(|f(x) - f_Q|)\,\mathrm{sgn}(f(x) - f_Q)}{|f(x) - f_Q|},$$

we see that the identity (2.9) and its variants lead to

$$\begin{aligned}\beta^{-1}\left(\sup_Q ⨍_Q \beta(|g(x)|)dx\right) &= \beta^{-1}\left(\sup_Q ⨍_Q \alpha(|f(x) - f_Q|)dx\right)\\ &= (\beta^{-1}\circ\alpha)(\|f\|_{BMO_\alpha})\\ &= \frac{\alpha(\|f\|_{BMO_\alpha})}{\|f\|_{BMO_\alpha}}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\|f\|_* &\geq \frac{\left|\sup_Q \fint_Q (f(x)-f_Q)g(x)dx\right|}{\beta^{-1}\left(\sup_Q \fint_Q \beta(|g(x)|)dx\right)} \\ &= \frac{\left|\sup_Q \fint_Q \alpha(|(f-f_Q)|)dx\right|}{\beta^{-1}\left(\sup_Q \fint_Q \beta(|g(x)|)dx\right)} \\ &= \|f\|_{BMO_\alpha}.\end{aligned}$$

In all, we get

$$\|f\|_{BMO_\alpha} \leq \|f\|_* \leq \hbar\|f\|_{BMO_\alpha}.$$

For any constant $k \geq 0$ and for $f \in BMO_\alpha(\mathbb{R}^d)$, the identity (2.14) yields

$$\|kf\|_{BMO_\alpha} \leq k\|f\|_* \leq k\hbar\|f\|_{BMO_\alpha}$$

and

$$\hbar\|kf\|_{BMO_\alpha} \geq k\|f\|_* \geq k\|f\|_{BMO_\alpha},$$

which imply the inequalities (2.12). When $\hbar = 1$, we have the homogeneity:

$$\|kf\|_{BMO_\alpha} = k\|f\|_{BMO_\alpha}.$$

This completes the proof. □

The functional $\|\cdot\|_{BMO_\alpha}$ on $BMO_\alpha(\mathbb{R}^d)$ may not produce a norm, since it does not always satisfy the homogeneity required for norms. Instead, by virtue of Minkowski's inequality (2.5), we may define a metric on $BMO_\alpha(\mathbb{R}^d)$ by

$$d(f,g) := \|f-g\|_{BMO_\alpha} \quad \text{for } f,g \in BMO_\alpha(\mathbb{R}^d).$$

It formulates a *complete* metric space on $BMO_\alpha(\mathbb{R}^d)$. The arguments comply the following standard procedure.

Suppose that $\{f_n\}$ is a Cauchy sequence in $BMO_\alpha(\mathbb{R}^d)$. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$d(f_{n_{k+1}}, f_{n_k}) \leq \frac{1}{(2\hbar)^k}, \quad k = 1, 2, \cdots.$$

Setting F with

$$F(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|,$$

we can notice that $|F(x)| < \infty$ *almost everywhere* $x \in \mathbb{R}^d$. *In fact, from the fact that*

$$\|F\|_{BMO_\alpha} \le \hbar \|f_{n_1}\|_{BMO_\alpha} + \sum_{k=1}^{\infty} \hbar^{k+1} \|f_{n_{k+1}} - f_{n_k}\|_{BMO_\alpha}$$
$$= \hbar \|f_{n_1}\|_{BMO_\alpha} + \hbar < \infty,$$

there exists a null set $N \subset \mathbb{R}^d$ *such that* $F(x) < \infty$ *for all* $x \in \mathbb{R}^d \setminus N$. Therefore for any $x \in \mathbb{R}^d \setminus N$, the absolute convergence of the series

$$f_{n_1}(x) + \sum_{k=1}^{\infty} [f_{n_{k+1}}(x) - f_{n_k}(x)]$$

makes it possible to define $f(x) := \lim_{k \to \infty} f_{n_k}(x)$ on $\mathbb{R}^d \setminus N$. The fact

$$\|f - f_{n_k}\|_{BMO_\alpha} = \left\| \sum_{j=k+1}^{\infty} f_{n_{j+1}} - f_{n_j} \right\|_{BMO_\alpha}$$
$$\le \sum_{j=k+1}^{\infty} \hbar^{j+1} \|f_{n_{j+1}} - f_{n_j}\|_{BMO_\alpha} = \frac{\hbar}{2^k}$$

and the inequality

$$\|f\|_{BMO_\alpha} \le \hbar \|f - f_{n_k}\|_{BMO_\alpha} + \hbar \|f_{n_k}\|_{BMO_\alpha}$$

yield $f \in BMO_\alpha$ and the convergence of $\{f_{n_k}\}$ to f in $BMO_\alpha(\mathbb{R}^d)$, which, in turn, implies the convergence of the original *Cauchy* sequence $\{f_n\}$ in $BMO_\alpha(\mathbb{R}^d)$.

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